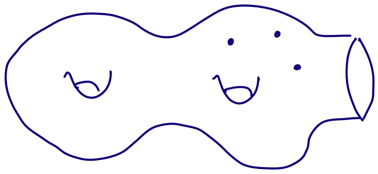
§1 Oriented surfaces $S = S^b_{g,n}$

genus g
n punctures / marked points
b boundary components



$S^1_{2,3}$

The Mapping Class Group

$$\mathrm{Mod}(S^b_{g,n}) \cong \mathrm{Diffeo}^+(S, \partial S) / \mathrm{Diffeo}_0(S, \partial S)$$

orientation-preserving diffeos mod isotopy

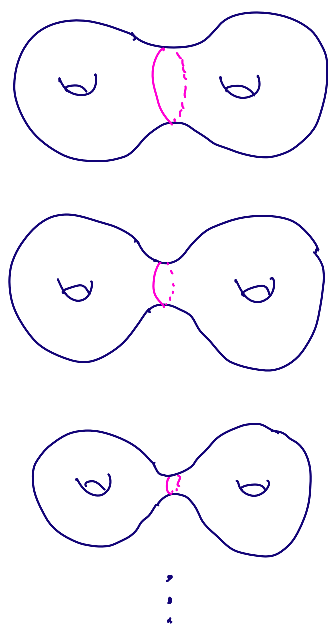
<u>$H^*(\mathrm{Mod}(S); \mathbb{Q})$</u> : H^* of the Moduli Space $\mathcal{M}^b_{g,n}$

$$\mathcal{M}^b_{g,n} = \underbrace{\mathrm{Teich}(S)}_{\substack{\text{parameterizes marked} \\ \text{hyperbolic structures} \\ \text{on } S}} / \mathrm{Mod}(S)$$

$$\mathrm{Teich}(S) \cong \mathbb{R}^d, \quad d = 6g - 6 + 3b + 2n$$

$\rightarrow$ $\mathcal{M}^b_{g,n}$ is not compact

Eg In $\mathrm{Teich}(S)$,



limit point does not exist in $\mathrm{Teich}(S)$

There is a bordification of Teich(S) s.t.

- $\overline{\text{Teich}(S)}$ is a manifold w/ bdry
- $\overline{\text{Teich}(S)} / \text{Mod}(S) \simeq \text{Teich}(S)/\text{Mod}(S)$

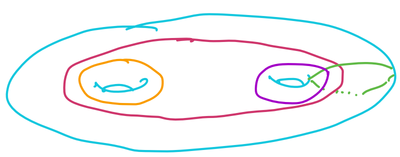
↑ compact

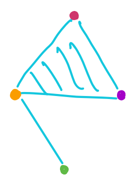
Turns out, the added border is $\simeq$ Curve Complex $C(S^b_{g,n})$

§2 The Curve Complex

k-simplices ⟷ $\underbrace{\text{(k+1) disjoint isotopy classes of simple closed curves}}_{\text{"curve system"}}$

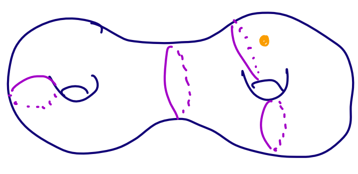
"curve system" ⟶ not allowing curves $\simeq$ pt or that are peripheral to a puncture or bdry component





<u>Note</u> : $C(S^b_{g,n}) \cong C(S_{g,n+b})$

- $\dim C(S_{g,n}) = 3g+n-4$ (need $3g+n-3$ curves in any pants decomposition)



<u>Thm (Harer '86)</u>

$$C(S_{g,n}) \simeq \bigvee_{\infty} S^m \qquad m = \begin{cases} n-4 & \text{if } g=0 \\ 2g-2 & \text{if } g \geq 1,\ n=0 \\ 2g-3+n & \text{if } g \geq 1,\ n \neq 0 \end{cases}$$

<u>Defn</u> : $\underset{\text{Steinberg module}}{St(S)} = St(S^b_{g,n}) := \tilde{H}_m(C(S^b_{g,n}))$

A Consequence :

Thm $H^{\nu-i}(\mathrm{Mod}(S);\mathbb{Q}) \cong H_i(\mathrm{Mod}(S); \mathrm{St}(S))$

$\nu = \nu(g,n,b)$

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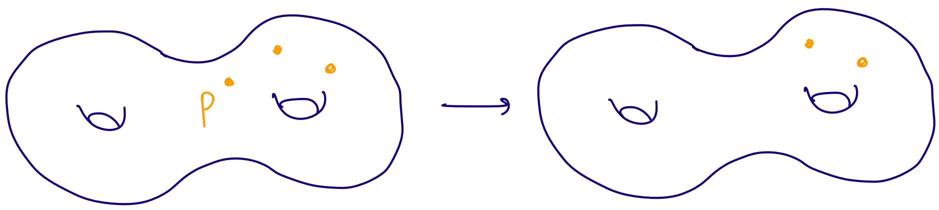
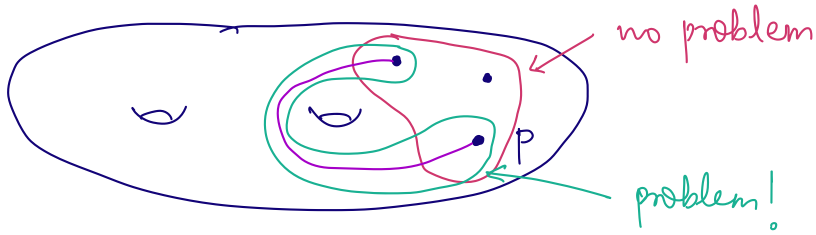
§ 3

"Forget a point" map $S_{g,n} \to S_{g,n-1}$



Analogue for $C(S_{g,n})$?



$X(S_{g,n})$ = subcomplex of $C(S_{g,n})$ spanned by 'good' curves

$\mathcal{A}_p$ = 'bad' curves

A for arcs

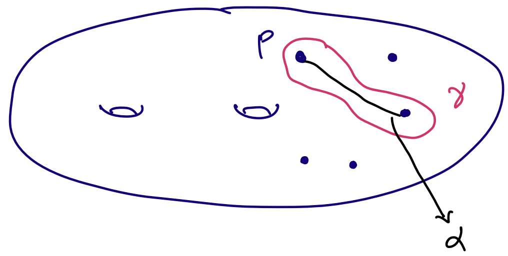
Have a map $X(S_{g,n}) \xrightarrow{f} C(S_{g,n-1})$

Prop^n (Harer, Brendle-Broaddus-Putman)

- $X(S_{g,n}) \xrightarrow{f} C(S_{g,n-1})$ is a htpy equiv.
- $C(S_{g,n}) \simeq \mathcal{A}_p * X(S_{g,n}) \simeq \mathcal{A}_p * C(S_{g,n-1})$

Can use to inductively show $C(S_{g,n}) \simeq \vee S^m$

Partial Proof



$\mathrm{Link}(\gamma) := \{ \text{simplices } \sigma \subset C(S_{g,n}) \mid \sigma * \gamma \text{ is a simplex of } C(S_{g,n}) \}$

Lk γ

<u>Note</u> : $Lk(\gamma) \subset \mathcal{X}(S_{g,n})$

<u>Claim</u> : $lk_{C(S_{g,n})}(\gamma) \xrightarrow{\ i\ } \mathcal{X}(S_{g,n})$ is a htpy equiv.

$$\begin{array}{c} \mathcal{X}(S_{g,n}) \\ \downarrow \pi \\ C(S_{g,n-1}) \end{array}$$

- $\pi \circ i$ is a simplicial iso
  $\Rightarrow i_*$ is injective

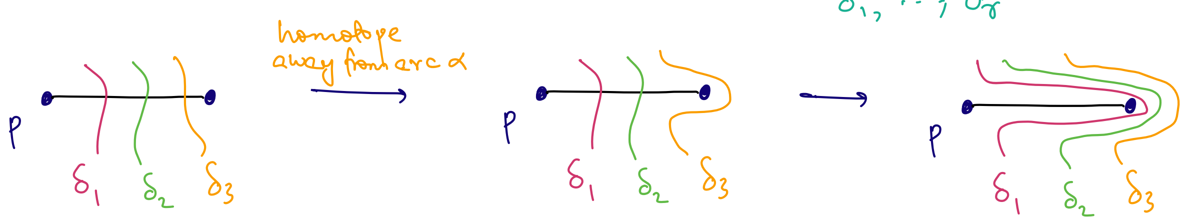
<u>Want to show</u> : i_* is surjective on homotopy

Fix k, simplicial structure on S^k.

Consider $\psi : S^k \to \mathcal{X}(S_{g,n-1})$

Want to homotope ψ s.t. $\psi(S^k) \cap \alpha = \phi$ ↙ defines γ

- Enough to do this on vertices $v_1, \dots, v_r$ of S^k
  ↳ represented by $\delta_1, \dots, \delta_r$

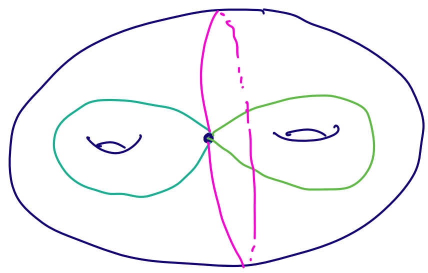


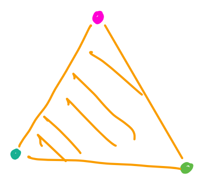
Hatcher Flow

§4 Broaddus' Resolution for $St(S_{g,1})$

A Variation of $C(S)$: The Arc Complex $\mathcal{A}(S)$

k-simplices $\leftrightarrow$ "arc systems"
k+1 arcs disjoint except at endpts

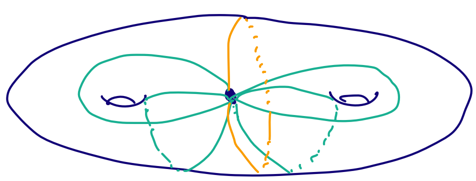




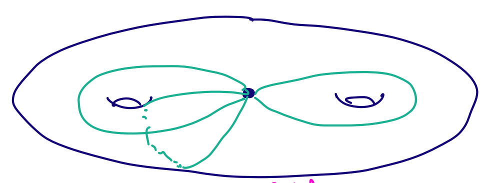
Fact : $\mathcal{A}(S) \simeq *$

(Can prove using the Hatcher flow idea)

The arc complex at infinity $\mathcal{A}_\infty$



filling



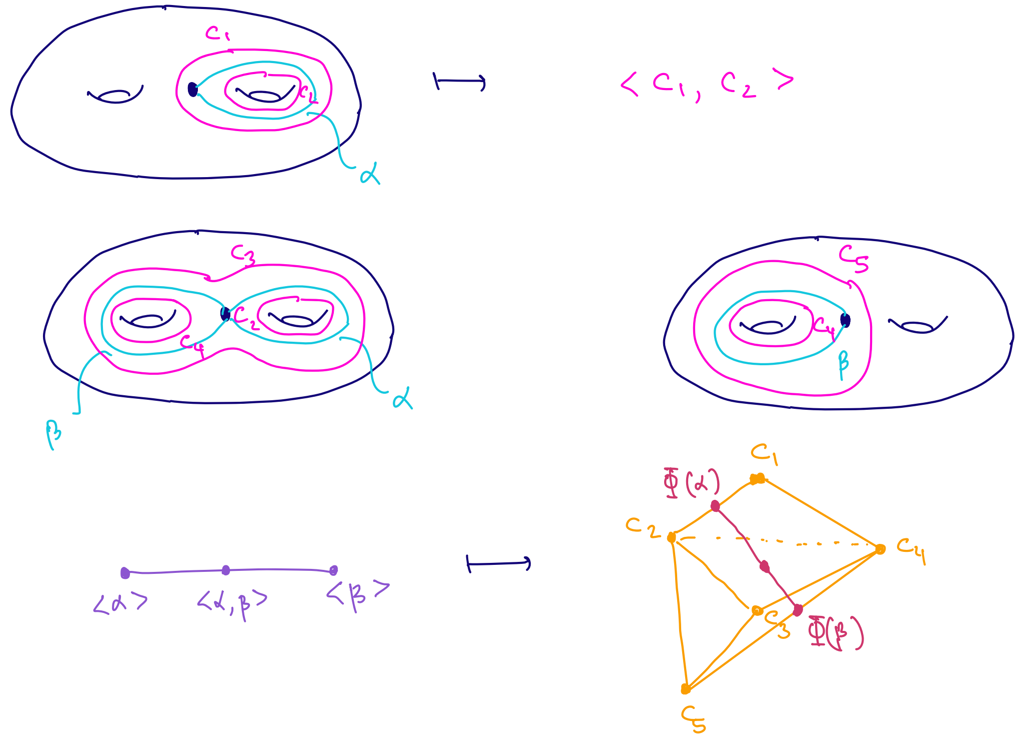
non-filling

A filling arc system : cuts $S_{g,n}$ into disks : 

(need $\geq 2g$ arcs)

$\mathcal{A}_\infty :=$ subcomplex of $\mathcal{A}$ spanned by non-filling systems

Prop^n : $\mathcal{A}_\infty(S) \simeq C(S)$



Thus:
- $H_*(\mathcal{A}, \mathcal{A}_\infty) = H_{*-1}(C(S)) = St_{g,1}$ for $* = 2g-1$
- $C_*(\mathcal{A}, \mathcal{A}_\infty) = 0$ for $* < 2g-1$

Setting $F_k = C_{2g+k-1}(\mathcal{A}, \mathcal{A}_\infty)$, get:

Prop^n (Broaddus)

$$0 \to F_{4g-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \to St(S_{g,1}) \to 0$$

is a $\mathrm{Mod}(S_{g,1})$-resolution for $St(S_{g,1})$ ← Is proj/flat over $\mathbb{Q}\mathrm{Mod}_{g,1}$, but not free

Cor: $St(S_{g,1}) \cong F_0 / \partial F_1$

↑ (oriented) 0-filling arc systems ($2g$ arcs)

↑ ordering of the arcs

Broaddus used this to show that $St(S_{g,1})$ is cyclic over $\mathrm{Mod}(S_{g,1})$.

Cor (Church-Farb-Putman) $H^{2g}(\mathrm{Mod}(S_{g,1}); \mathbb{Q}) = H_0(\mathrm{Mod}(S_{g,1}); St(S_{g,1}))$
$= 0$